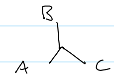
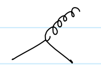
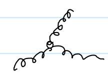
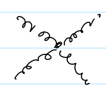
Now for the <u>Weak Interactions</u>

Let's compare:

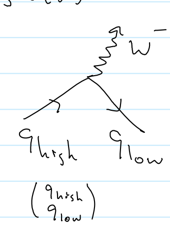
	External State Labels	Internal Propagators	Vertex Factors
ABC	none	$\frac{i}{q^2-m^2c^2}$	 $-ig$
QED	$u, \bar{u}, v, \bar{v}$ $\epsilon_\mu, \epsilon^*_\mu$	$\frac{i(\not q + mc)}{q^2-m^2c^2}$ $-\frac{ig_{\mu\nu}}{q^2}$	 $ig_e\gamma^\mu$
QCD	$uc, \bar{u}c^\dagger, vc, \bar{v}c^\dagger$ $\epsilon_\mu a^\alpha, \epsilon^*_\mu a^{\alpha *}$	$\frac{i(\not q + mc)}{q^2-m^2c^2}$ $-\frac{ig_{\mu\nu}\delta^{\alpha\beta}}{q^2}$	 $-\frac{ig_s}{2}\lambda^\alpha\gamma^\mu$  "horrible" (8.43)  "horribler" (8.44)
Weak	$u, \bar{u}, v, \bar{v}$ $\underbrace{\epsilon_\mu, \epsilon^*_\mu}$ Now have 3-component polarizations since $W^\pm, Z^0$ are <u>massive</u> vector particles.	$\frac{i(\not q + mc)}{q^2-m^2c^2}$ $\underbrace{\frac{-i(g_{\mu\nu}-q_\mu q_\nu/m^2c^2)}{q^2-m^2c^2}}$ Note this doesn't limit to $m=0$ propagator, but that is because # of d.o.f. change discontinuously $(3\to 2)$.	 $-\frac{ig_w}{2\sqrt{2}}\underbrace{\gamma^\mu(1-\gamma^5)}_{V-A}$  $-\frac{ig_z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5)$

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e^-, μ^-, τ^-	$-\frac{1}{2}+2\sin^2\theta_w$	$-\frac{1}{2}$
u, c, t	$\frac{1}{2}-\frac{4}{3}\sin^2\theta_w$	$\frac{1}{2}$
d, s, b	$-\frac{1}{2}+\frac{2}{3}\sin^2\theta_w$	$-\frac{1}{2}$

Note: $W^\pm$ are anti-partners
Z^0 is its own anti-particle (like γ)

Also recall that Strong + Weak operators don't commute, so they have different eigenstates.
Quarks are usually created in Strong force eigenstates, but decay by Weak force (so in weak eigenstates).

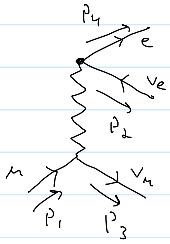
$$\begin{array}{c}\text{high}\\ \text{low}\end{array} \begin{pmatrix} u \\ d \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix}\begin{pmatrix} t \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ d' \end{pmatrix}\begin{pmatrix} c \\ s' \end{pmatrix}\begin{pmatrix} t \\ b' \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}\begin{pmatrix} d \\ s \\ b \end{pmatrix}$$



$$-\frac{ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)V_{high-low}$$

One of the important things about the Weak interactions ($W^\pm$ to be specific) is that they provide the mechanism for true particle decay (not annihilation). This leads to useful lifetime results.

<u>Muon decay</u> $\mu \to \nu_\mu + e + \bar{\nu}_e$



$$\Rightarrow \quad \mathcal{M} = \bar{u}(3)\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)u(1)\underbrace{\frac{-i(g_{\mu\nu}-q_\mu q_\nu/M_w^2c^2)}{q^2-M_w^2c^2}}_{\substack{\text{If } q \ll M_w c \text{ (typical for low-energy muons)} \\ \approx \frac{ig_{\mu\nu}}{M_w^2c^2}}}\bar{u}(4)\frac{-ig_w}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)v(2)$$

$$\mathcal{M} \approx \frac{g_w^2}{8M_w^2c^2}\,\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\bar{u}(4)\gamma_\nu(1-\gamma^5)v(2)$$

As usual we average over incoming spins and sum over final spins using Casimir's trick and the γ matrix trace theorems:

$$\langle|\mathcal{M}|^2\rangle_1 = 2\left(\frac{g_w}{M_w c}\right)^4(p_1\cdot p_2)(p_3\cdot p_4)$$

To evaluate the muon lifetime as seen in its rest frame we use $p_1 = (m_\mu c, \vec{0})$ to obtain (after some work):

$$\langle|\mathcal{M}|^2\rangle_1 = \left(\frac{g_w}{M_w c}\right)^4 m_\mu^2 E_2(m_\mu c^2 - 2E_2)$$

This can be used in Fermi's Golden Rule for decays (the full integral form, not the simple 2-body decay form!) and working w/ $m_e \simeq 0$ since $m_e c^2$ is a small percentage of the energy released $(m_\mu - m_e)c^2$.
$0.5\ \frac{MeV}{c^2}$ $105\ \frac{MeV}{c^2}$

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle_1}{2\hbar m_\mu}\frac{d^3p_2}{(2\pi)^3 2|\vec{p}_2|}\frac{d^3\vec{p}_3}{(2\pi)^3 2|\vec{p}_3|}\frac{d^3p_4}{(2\pi)^3 2|\vec{p}_4|}(2\pi)^4\delta^4(p_1-p_2-p_3-p_4)$$

The δ function can be used to evaluate $\int d^3\vec{p}_3$ and the remaining two momenta are bound to satisfy:

$|\vec{p}_2| < \frac{1}{2}m_\mu c$ max when $2 \longleftarrow \quad \longrightarrow 3, 4$ (decreases as ...)

$|\vec{p}_4| < \frac{1}{2}m_\mu c$ max when $4 \longleftarrow \quad \longrightarrow 2, 3$

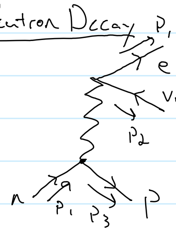
$|\vec{p}_2| + |\vec{p}_4| > \frac{1}{2}m_\mu c$ min when $3 \longleftarrow \quad \longrightarrow 2, 4$ (increases as $3 \longleftarrow$ 2, 4)

After more work: $\tau = \frac{1}{\Gamma} = \left(\frac{M_w}{m_\mu g_w}\right)^4\frac{12\hbar(8\pi)^3}{m_\mu c^2}$

Using the observed muon lifetime 2.1970×10^{-6} s and $M_w = 80{,}420\ \frac{MeV}{c^2}$ $\Rightarrow g_w = 0.653 \Rightarrow \alpha_w = \frac{g_w^2}{4\pi} = \frac{1}{29.5}$

Compare to $\alpha = \frac{1}{137}$ for QED! The weak force isn't <u>weak</u>, it's just that $W^\pm$ are so heavy!

Neutron Decay $\quad n \to p + e + \bar{\nu}_e$



$$\Rightarrow \langle |M|^2 \rangle_1 = 2\left(\frac{g_w}{m_w c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Note: Same expression as in muon decay!

This time when employing the Golden Rule, we cannot ignore the contribution from m_e to the energy released $(m_n - m_p - m_e)c^2$. This complicates the calculation considerably!

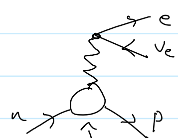
m_n: $939.6\ \frac{MeV}{c^2}$, m_p: $938.3\ \frac{MeV}{c^2}$, m_e: $0.5\ \frac{MeV}{c^2}$

After much work and some approximations: $\tau = \frac{1}{\Gamma} \simeq 1318\ s$ (using g_w found from muon decay)

But experimentally $\tau = 885.7\ \frac{MeV}{c^2}$

To be fair, it is hard to say just how W couples to the n or p since they are a "hot $\overset{QCD}{\vee}$ mess" of quarks and virtual gluons and virtual quark-anti-quark pairs! This is not an issue when γ couples to the hot QCD mess since even in a mess of virtual pairs, the net electric charge is conserved. There is no theoretical starting point for "weak charge" conservation. So a similar calculation w/ QED gets a much better answer.

To quantify what is happening we say



This modified weak vertex can be written w/ $(1-\gamma^5) \to (c_V - c_A\gamma^5)$

Experimentally $c_V = 1$, $c_A = 1.27$ (determined by the same decay but within nuclei, e.g. $^{14}O \to {}^{14}N$).

Partially conserved axial current

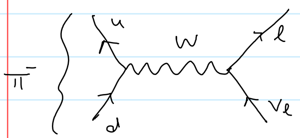
Weak vector charge conservation (CVC hypothesis $\Rightarrow$ some symmetry protecting it)

CVC: conserved vector current

Somehow all the craziness inside the n+p does not change this part!

With this correction $\tau = 901s$ with other corrections expected to bring it to the measured value.

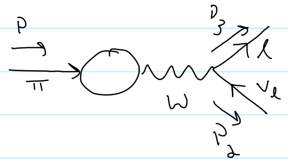
Pion decay $\pi^- \to \ell + \nu_\ell$



Actually more of a "scattering" process but the answer is interesting.

To calculate the actual π^- lifetime is challenging since it actually depends on the ground state wavefunction of the $d\bar{u}$ system. However what can be independently calculated is the ratio of $\pi^- \to e + \bar{\nu}_e$ to $\pi^- \to \mu + \bar{\nu}_\mu$.

To do this we use



$$\Rightarrow \mathcal{M} = \frac{g_w^2}{8(M_W c)^2}\,\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)\,F^\mu$$

F encodes ignorance of $\bigcirc$

Now F^μ must depend on some initial 4-vector, but the only one is $p^\mu \Rightarrow F^\mu = \underset{\text{scalar}}{f_\pi}\, p^\mu$

But f_π must depend on the only scalar available $p^2 = m_\pi^2 c^2$. This means that the result for F^μ is independent of which $\ell, \bar{\nu}_\ell$ we use!

Then $\langle|\mathcal{M}|^2\rangle = \frac{1}{8}\left[f_\pi\left(\frac{g_w}{M_W c}\right)^2\right]^2\left[2(p\cdot p_2)(p\cdot p_3) - p^2(p_2\cdot p_3)\right]$

This time we can use the 2-body decay formula:

$$\Gamma = \frac{|\vec{p}_2|}{8\pi\hbar m_\pi^2 c}\langle|\mathcal{M}|^2\rangle$$

Since $|\vec{p}_2| = \frac{c}{2m_\pi}(m_\pi^2 - m_\ell^2)$

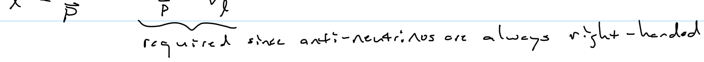
$$\Gamma = \frac{f_\pi^2}{\pi\hbar m_\pi^3}\left(\frac{g_w}{4M_W}\right)^4 m_\ell^2(m_\pi^2 - m_\ell^2)^2$$

We can't evaluate Γ w/out f_π, but we can evaluate:

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 1.283 \times 10^{-4}$$

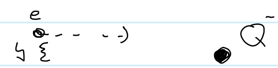
This should be surprising since we normally think that the kinematic likelihood is driven by mass differences and hence $\pi \to e + \bar{\nu}_e$ should be more likely than $\pi \to \mu + \bar{\nu}_\mu$.

What is going on? Well π^- is spin-0 so when it decays ℓ and ν_ℓ must come out w/ equal helicities (or opposite spins).



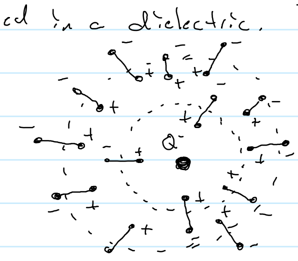
But m_e is "almost" zero so it is almost always produced as a left-handed only particle whereas m_μ is very massive and so happy to exist as a right-handed particle.

A Classical Fairy Tale

Consider a massive particle with unknown charge happily at rest: 

If we wanted to measure the charge, we might scatter a known charge with known p^μ and use the observed trajectory to infer q_μ. If we used an electron, we could even work in the limit that the particle remains at rest the whole time.
If we knew the impact parameter b we could use: $Q^- = \frac{b\,2E}{q_e \cot(\frac{\theta}{2})}$

Now suppose the particle is submersed in a dielectric. Then the negatively charged particle will gently reorient the dipoles creating:



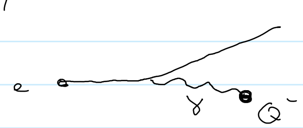
The "effective charge" due to dipole screening. For the inner Gaussian surface we have $Q^- + 4e$, while for the outer we have $Q^- + 9e$. Large distances $\Rightarrow$ more screening.

Now the issue is that if we send in a probe electron, it will only respond to the effective charge which varies w/ distance. If we call the distance of closest approach $\tilde{r}$, then the largest (screened) value seen would be $Q^-(\tilde{r})$. But how close the electron comes $(\tilde{r})$ is determined by its momentum (p) so this could be rewritten as $Q^-_\mu(p)$.

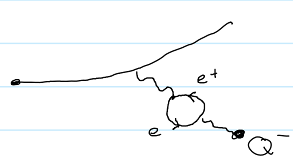
So in this classical example we see a use for effective charges which vary w/ the momentum scale of the experiment. Of course we can always seek the classical "true" μ charge by either probing w/ p high enough that we get closer to μ than any screening pair, or simpler still we just remove μ from the dielectric.

A Quantum Fairy Tale

Now suppose we have a Q^- in free space and we want to probe it w/ an e. We know that roughly:



and we replace definite results w/ probabilities.

However we know that we could also have: 

and the e^+e^- pair could act like a dielectric.

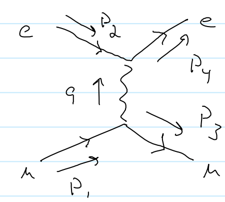
This turns out to be a crucial part of the story of QFT and particle physics, and will lead to the process of renormalization.

Renormalization

We want to explore the "screening" of real electric charge by virtual particle pairs in a more quantitative way. This will also lead us to understand how to handle many divergences encountered in Feynman amplitudes.

Consider $e^- + \mu^- \to e^- + \mu^-$

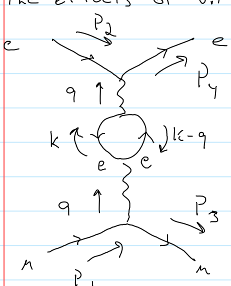
At 2nd order we have the "tree-level" contribution (simplest nontrivial):



$$\Rightarrow \mathcal{M}_{tree} = -g_e^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] \frac{\eta_{\mu\nu}}{q^2} \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

$\boxed{q \equiv p_2 - p_4}$

The effects of virtual particle pairs start at 4th order with the most important contribution being:



We will later learn why the other 4th order contributions do not play a role in what is to come.
To evaluate a diagram like this with a purely internal loop of matter we need the Feynman rule: For internal loops of matter write down the ordered product of vertex factors and propagators, the take the trace (in spin space) and $\times(-1)$

naively expect $\int^\infty \frac{k^3 dk\, k^2}{k^4} = \int^\infty k\,dk = k^2\Big|^\infty \to \infty$ but actually only $\ln k\Big|^\infty$

$$\Rightarrow \mathcal{M}_{1\text{-loop}} = -g_e^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] i g_e^2 \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu(\not{k}+m_e c)\gamma_\nu(\not{k}-\not{q}+m_e c)\right]}{q^2(k^2-m_e^2c^2)\left[(k-q)^2-m_e^2c^2\right]}} \left[\bar{u}(4)\gamma^\nu u(2)\right]$$

Call this $-i\underbrace{\eta_{\mu\nu}}_{} I(q^2)$
due to $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$

After some hell of a lot of massaging: $I(q^2) = -\frac{1}{12\pi^2}\left\{\underbrace{\int_{m_e^2}^{\infty}\frac{dz}{z}}_{\text{blows up as } \ln z} - \underbrace{6\int_0^1 z(1-z)\ln\left[1-\frac{q^2}{m_e^2c^2}z(1-z)\right]dz}_{\text{finite } f\left(\frac{-q^2}{m_e^2c^2}\right)}\right\}$

finite $f\left(\frac{-q^2}{m_e^2c^2}\right)$ w/ $f(-q^2=0)=0$ and $f(-q^2)$ increasing w/ $|q^2|$

$q^2 < 0$ for this process!

The next step in the "renormalization" program is to "regularize" the divergence (make it a finite contribution). We will simply use an upper "cuttoff":
$$\int_{m_e^2}^{M_c^2}\frac{dz}{z} = \ln\left(\frac{M_c^2}{m_e^2}\right)$$

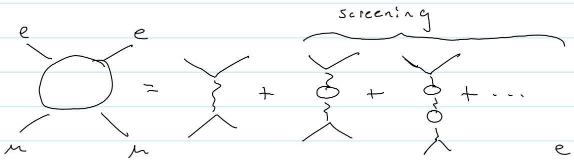
Then: $I(q^2) = -\frac{1}{12\pi^2}\left\{\ln\frac{M_c^2}{m_e^2} - f\left(\frac{-q^2}{m_e^2c^2}\right)\right\}$

notice that divergence is still present when $M_c \to \infty$

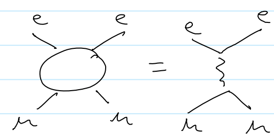
And: $\mathcal{M}_{tree} + \mathcal{M}_{1\text{-loop}} = -g_e^2\left[\bar{u}(3)\gamma^\mu u(1)\right]\frac{\eta_{\mu\nu}}{q^2}\left(1 - \frac{g_e^2}{12\pi^2}\left\{\ln\frac{M_c^2}{m_e^2} - f\left(\frac{-q^2}{m_e^2c^2}\right)\right\}\right)\left[\bar{u}(4)\gamma^\nu u(2)\right]$

So how do we interpret this?

We measure:



But interpret our measurement "classically", i.e.



so we really measure an "effective" charge that includes all loop contributions

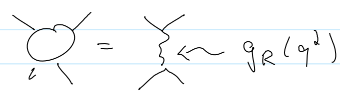
To formalize this we simply rewrite our results in terms of the effective or "renormalized" charge/coupling:

$$g_R(q^2) \equiv g_e\sqrt{1 - \frac{g_e^2}{12\pi^2}\left[\ln\left(\frac{M_c^2}{m_e^2}\right) - f\left(\frac{-q^2}{m_e^2c^2}\right)\right] + \cdots} = \text{physically measured value of } e\sqrt{\frac{4\pi}{\hbar c}} \text{ in experiment at momentum transfer scale } q = p_2 - p_4$$

Then of course: $\mathcal{M}_{tree} + \mathcal{M}_{1\text{-loop}} = -g_R(q^2)\left[\bar{u}(3)\gamma^\mu u(1)\right]\frac{g_{\mu\nu}}{q^2}\left[\bar{u}(4)\gamma^\nu u(2)\right]$

which looks like it came from  alone!

This is an example of an "effective" theory where the quantum loop corrections are bundled into renormalized quantities and the result is interpreted as the "classical" tree-level, i.e.



This is often written in terms of the zero-momentum (large distance) value: $g_R(0) = g_e\sqrt{1 - \frac{g_e^2}{12\pi^2}\ln\left(\frac{M_c^2}{m_e^2}\right)}$ to obtain:

$$g_R(q^2) = g_R(0)\sqrt{1 + \frac{g_R^2(0)}{12\pi^2} f\left(\frac{-q^2}{12\pi^2}\right)}$$

So lets compare this to the screening story. If we are very far away ($q^2 \sim 0$) $\Rightarrow$ $g_R(q^2) = g_R(0)$. As we probe closer, i.e. $|q^2| > 0$ and since $f\left(-\frac{q^2}{12\pi^2}\right)$ increases we find that: $g_R(q^2 \neq 0) > g_R(0)$, i.e. we had screening!

Okay fine, but what about the divergent $\ln\left(\frac{M_c^2}{m_e^2}\right)$ term in $g_R(0)$? Isn't it causing everything to blow up? Well it depends on your input. We built our fundamental theory w/ a coupling $g_e = e\sqrt{\frac{4\pi}{\hbar c}}$ but what is the fundamental (or "bare" or "unscreened") value of e? Unlike in the classical example, we cannot simply remove the μ from this "dielectric bath". Worse still, to use higher $|q^2|$ to probe beyond any screening actually requires $|q^2| \to \infty$ which we simply cannot do!

Are we lost? No, not really. After all, we can (and do) measure $g_R(0)$ itself (in fact that is the value you see in tables), and then we can just work in terms of $g_R(0)$ instead of the unknowable g_e. The upshot is that working in terms of $g_R(0)$ means the problematic infinities no longer appear!

Three important points:

1. In principle this should be done to all orders (# of loops) to get the correct $g_R(q^2)$, but lower # dominates.
2. We encountered ∞'s, but now realize that this was due to erroneously using the measured electron charge where we should be using the, unknown, bare value. This situation crops up in many QFTs, and when we can "fix" it like this, we call the theory renormalizable. The standard model is, but perturbative quantum gravity is not.
3. Even w/out the problem of ∞'s, we see the need for renormalizing because the truth is that measured values of "constant" actually depend on q, i.e. they run (running couplings).

Often people will leave out the q^2 dependence and work in terms of $g_R(0)$. But then they must include all loop corrections. But these will now be finite since they are in terms of $g_R(0)$ and not g_e!